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PLANE WAVES IN SIMPLE ELASTIC SOLIDS AND DISCONTINUOUS DEPENDENCE OF SOLUTION ON BOUNDARY CONDITIONS

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Abstract—Using stress as the dependent variable instead of the deformation gradient, plane waves of finite amplitude in simple elastic solids are studied. For isotropic materials there are two plane polarized simple waves as well as shock waves and one circularly polarized simple wave which can also be regarded as a shock wave. With the aid of the stress paths for simple waves and shock waves in the stress space introduced here, one can see clearly what combination of simple waves and/or shock waves is needed to satisfy the initial and boundary conditions. We use second order isotropic hyperelastic materials to illustrate the ideas. In one example we show that the solution requires as many as four simple waves. In another we show that depending on the boundary condition there are more than eight possible solutions to the problem. We also present an example in which the solution does not depend continuously on the boundary condition. This implies that in experiments if the applied load at the boundary is not properly controlled, any slight deviation in the applied load would result in a finite different response in the material.

1. INTRODUCTION

Plane finite amplitude waves in simple elastic solids have been studied by many investigators [1–7]. The deformation gradient was invariably used as one of the dependent variables. In [4] Bland showed that there existed three plane simple waves and three associated plane shock waves in isotropic hyperelastic materials; two of the simple waves and two of the shock waves are plane polarized with respect to the deformation-gradient space and the remaining simple wave and shock wave are circularly polarized. Davison[5] obtained centered simple wave solutions to some initial and boundary conditions by a semi-inverse approach, i.e. one assumed a certain combination of simple waves and/or shock waves to see what deformation gradients should be prescribed as the initial and boundary conditions.

One of the main purposes of this paper is to present a means of determining the correct combination of simple waves and/or shock waves to satisfy the prescribed initial and boundary conditions. Since deformation gradients are seldom prescribed as the initial and boundary conditions, we use stress as the dependent variable. We introduce the "stress paths" for simple waves and "stress paths" for shock waves to find the solutions. The idea of using the stress paths for simple waves in solving the problem was first employed in the study of elastic–plastic wave propagation [8, 9]. However, stress paths for shock wave were straight lines because the shock wave studied in [8, 9] was linear and involved only one stress component.

After presenting the basic equations for plane waves in general simple elastic solids and theory of simple waves in Section 2, we consider in Section 3 the constitutive equations for general isotropic elastic solids which are undergoing a plane wave motion. Only two response functions of two arguments are necessary to analyze the plane wave motion. Basic results for simple waves in general isotropic elastic solids are then derived in Section 4. It is shown that with stress as the dependent variable, we also have two plane polarized and one circularly polarized simple waves. The general results are specialized for second order isotropic hyperelastic materials in Section 5. We use the latter to illustrate the stress paths for simple waves and show how one can find the right combination of the simple waves to satisfy certain initial and boundary conditions. We show that for some problems the solution consists of four simple waves.

Since not all initial and boundary value problems can be solved by combining the simple waves, we study in Section 6 plane shock waves in general isotropic materials. The results are

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again specialized for second order materials in Section 7 which are then used as illustrative examples in Section 8. We illustrate how one can find the correct solution by using the stress paths for simple waves and shock waves. We also present an example in which the solution does not depend continuously on the boundary conditions.

2. BASIC EQUATIONS FOR PLANE WAVES AND THEORY OF SIMPLE WAVES

In a fixed rectangular coordinate system, let x_i and X_i be, respectively, the position of a particle at time t and at $t = 0$ which is taken as the undeformed state. The material occupies the half space $X_1 \geq 0$. Assuming that the plane wave is propagating in the X_1-direction, we have

$$x_i = X_i + u_i(X, t) \tag{1}$$

where $X = X_1$ and u_i is the displacement. The deformation gradient $\mathbf{F}$ then has the expression

$$F_{ij} = \frac{\partial x_i}{\partial X_j} = \begin{bmatrix} 1+p_1 & 0 & 0 \\ p_2 & 1 & 0 \\ p_3 & 0 & 1 \end{bmatrix} \tag{2}$$

$$p_i = \partial u_i / \partial X. \tag{3}$$

Let $\mathbf{T}$ be the Piola–Kirchhoff stress tensor of the first kind. For simple elastic materials $\mathbf{T}$ is a given function of $\mathbf{F}$

$$\mathbf{T} = \mathbf{T}(\mathbf{F}). \tag{4}$$

Since $\mathbf{F}$ is a function of X and t, so is $\mathbf{T}$. The equations of motion and the continuity of displacement can then be written as

$$\frac{\partial s_i}{\partial X} - \rho_0 \frac{\partial v_i}{\partial t} = 0 \tag{5}$$

$$\frac{\partial v_i}{\partial X} - \frac{\partial p_i}{\partial t} = 0 \tag{6}$$

where ρ_0 is the mass density in the undeformed state and

$$v_i = \frac{\partial u_i}{\partial t} \tag{7}$$

$$s_i = T_{i1} = \sigma_{i1}. \tag{8}$$

In eqn (8) σ_{ij} is the Cauchy stress which is related to T_{ij} by [10]

$$\boldsymbol{\sigma} = J^{-1}\mathbf{T}\mathbf{F}^T \tag{9}$$

where $J = \|\mathbf{F}\|$ and the superscript T stands for the transpose. The second equality of eqn (8) follows from the special form of $\mathbf{F}$ given in eqn (2).

Noticing that s_i is a function of p_1, p_2 and p_3

$$s_i = s_i(p_1, p_2, p_3) \tag{10}$$

and assuming that its inverse as well as the derivatives

$$G_{ij} = \partial p_i / \partial s_j \tag{11}$$

exist, we write eqns (5), (6) in matrix notations as

$$\left.\begin{array}{l} \mathbf{s}_{,X} - \rho_0 \mathbf{v}_{,t} = \mathbf{0} \\ \mathbf{v}_{,X} - \mathbf{G}\mathbf{s}_{,t} = \mathbf{0} \end{array}\right\} \tag{12}$$

where a comma stands for the partial differentiation. Introducing the following matrices

$$\mathbf{A} = \begin{bmatrix} \rho_0 \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{G} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \mathbf{0} & -\mathbf{I} \\ -\mathbf{I} & \mathbf{0} \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} \mathbf{v} \\ \mathbf{s} \end{bmatrix} \tag{13}$$

where $\mathbf{I}$ is the unit matrix, eqn (12) has the form

$$\mathbf{A}\mathbf{w}_{,t} + \mathbf{B}\mathbf{w}_{,X} = \mathbf{0}. \tag{14}$$

The formulation here differs from that of [1–7] in that the dependent variables are $\mathbf{s}$ and $\mathbf{v}$, not $\mathbf{p}$ and $\mathbf{v}$. Notice that $\mathbf{B}$ is a constant matrix while $\mathbf{A}$ depends on $\mathbf{s}$ only.

To find a simple wave solution for eqn (14), we assume that $\mathbf{w} = \mathbf{w}(\phi)$ where $\phi = \phi(X, t)$. Equation (14) then reduces to

$$(c\mathbf{A} - \mathbf{B})\frac{\mathrm{d}\mathbf{w}}{\mathrm{d}\phi} = \mathbf{0} \tag{15}$$

$$c = -\phi_{,t}/\phi_{,X}. \tag{16}$$

A nontrivial solution for $\mathrm{d}\mathbf{w}/\mathrm{d}\phi$ exists if

$$\|c\mathbf{A} - \mathbf{B}\| = 0 \tag{17}$$

where c can be identified as the characteristic wave speeds. Along the line $\phi(X, t) = \text{constant}$, $\mathbf{w}$ and hence $\mathbf{s}$ is constant. Equation (17) implies that c also is constant. It follows from eqn (16) that $\phi(X, t) = \text{constant}$ are straight lines in the (X, t) plane. These straight lines need not pass through the origin [9]. If they do, they are called centered simple waves.

In view of eqn (13), eqn (17) is equivalent to

$$\|\mathbf{G} - \eta\mathbf{I}\| = 0 \tag{18}$$

$$\eta = (\rho_0 c^2)^{-1} \tag{19}$$

If $\mathbf{r} = (r_1, r_2, r_3)$ is the right eigenvector of $(\mathbf{G} - \eta\mathbf{I})$,

$$(\mathbf{G} - \eta\mathbf{I})\mathbf{r} = \mathbf{0} \tag{20}$$

it can be shown that the right eigenvector of $(c\mathbf{A} - \mathbf{B})$ is $(-\mathbf{r}/\rho_0 c, \mathbf{r})$. Since eqn (15) implies that $\mathrm{d}\mathbf{w}/\mathrm{d}\phi$ is proportional to the right eigenvector of $(c\mathbf{A} - \mathbf{B})$, we obtain [11, 12]

$$\frac{\mathrm{d}s_1}{r_1} = \frac{\mathrm{d}s_2}{r_2} = \frac{\mathrm{d}s_3}{r_3} \tag{21a}$$

$$\mathrm{d}v_i = -\mathrm{d}s_i/\rho_0 c. \tag{21b}$$

Noticing that $\mathbf{r}$ is a function of $\mathbf{s}$, eqn (21a) can be integrated to provide two-parameter families of curves in the (s_1, s_2, s_3) space. These curves are called "stress paths" for simple waves. Since there are three eigenvectors $\mathbf{r}$ associated with three eigenvalues η, there are three sets of "stress paths" for simple waves.

In the rest of the paper, we will limit our attention to isotropic simple elastic materials.

3. ISOTROPIC SIMPLE ELASTIC SOLIDS

For isotropic simple elastic solids, the constitutive equation for the Cauchy stress $\boldsymbol{\sigma}$ can be written as [10]

$$\boldsymbol{\sigma} = J_0\mathbf{I} + J_1(\mathbf{FF}^T) + J_2(\mathbf{FF}^T)^2 \tag{22}$$

where $\mathbf{FF}^T$ is the left Cauchy–Green tensor, J_i $(i = 0, 1, 2)$ are functions of the invariants of $(\mathbf{FF}^T)$. For the $\mathbf{F}$ given by eqn (2), it can be shown that J_i are functions of p_1 and $p_2^2 + p_3^2$ only. Substitution of $\mathbf{F}$ into eqn (22) and noticing that $s_i = \sigma_{i1}$ from eqn (8), we obtain

$$\left.\begin{aligned} s_1 &= f(\epsilon, \gamma^2) \\ s_2 &= p_2 g(\epsilon, \gamma^2) \\ s_3 &= p_3 g(\epsilon, \gamma^2) \end{aligned}\right\} \tag{23}$$

where

$$p_1 = \epsilon,\ \ p_2 = \gamma\cos\theta,\ \ p_3 = \gamma\sin\theta \tag{24}$$

and f and g are arbitrary functions of ϵ and γ^2. ϵ and γ may be identified as the longitudinal and shear strain, respectively. For hyperelastic materials, we must have

$$\frac{\partial s_i}{\partial p_j} = \frac{\partial s_j}{\partial p_i} \tag{25}$$

which implies that

$$g_{,\epsilon} = 2f_{,\gamma^2}. \tag{26}$$

Again, a comma stands for the partial derivative.

It is not difficult to see that the inverse of eqn (23) can be written as

$$\left.\begin{aligned} p_1 &= \epsilon = h(\sigma, \tau^2) \\ p_2 &= s_2 q(\sigma, \tau^2) \\ p_3 &= s_3 q(\sigma, \tau^2) \end{aligned}\right\} \tag{27}$$

where

$$s_1 = \sigma, s_2 = \tau\cos\theta, s_3 = \tau\sin\theta \tag{28}$$

and h and q are arbitrary functions of σ and τ^2. σ and τ may be identified as the normal and shear stress, respectively. θ is the angle the shear stress τ makes with X_2-axis. If the material is hyperelastic, $G_{ij} = G_{ji}$ and it follows from eqns (11) and (27) that

$$q_{,\sigma} = 2h_{,\tau^2}. \tag{29}$$

4. SIMPLE WAVES IN GENERAL ISOTROPIC MATERIALS

With p_i as functions of s_i given by eqn (27), substitution of eqn (11) into eqn (18) yields the following cubic equations for η

$$(\eta - q)\{\eta^2 - (h_{,\sigma} + q + 2\tau^2 q_{,\tau^2})\eta + qh_{,\sigma} + 2\tau^2(h_{,\sigma}q_{,\tau^2} - h_{,\tau^2}q_{,\sigma})\} = 0. \tag{30}$$

Denoting the three roots of η by η_1, η_2, η_3, *we have*

$$\left.\begin{aligned}\eta_1 &= \frac{1}{2}(h_{,\sigma} + q + 2\tau^2 q_{,\tau^2}) - Y\\ \eta_2 &= q\\ \eta_3 &= \frac{1}{2}(h_{,\sigma} + q + 2\tau^2 q_{,\tau^2}) + Y\end{aligned}\right\} \tag{31}$$

where

$$Y = \frac{1}{2}\{(-h_{,\sigma} + q + 2\tau^2 q_{,\tau^2})^2 + 8\tau^2 h_{,\tau^2} q_{,\sigma}\}^{1/2}. \tag{32}$$

The reason for ordering $\eta_i (i = 1, 2, 3)$ in the way shown in eqn (31) is that for some materials (see Section 5) $\eta_1 \leq \eta_2 \leq \eta_3$ and hence $c_3 \leq c_2 \leq c_1$.

We will now discuss the "stress paths" for simple waves for each wave speed separately.

(i) $\eta = \eta_2 = q$, $(c = c_2)$

The right eigenvector $\mathbf{r}$ of eqn (20) for $\eta = \eta_2$ is

$$\mathbf{r} = (0, s_3, -s_2) \tag{33}$$

and eqn (21a) yields the following solution

$$\left.\begin{aligned} s_1 &= \text{constant}\\ s_2^2 + s_3^2 &= \text{constant}\end{aligned}\right\} \tag{34}$$

or in view of eqn (28)

$$\left.\begin{aligned} \sigma &= \text{constant}\\ \tau &= \text{constant}\end{aligned}\right\} \tag{35}$$

In the (s_1, s_2, s_3) space, the "stress paths" given by eqn (34) are circles. Following Bland[4], we may call the simple wave associated with $c = c_2$ the "circularly polarized" simple wave although Bland used the deformation gradient instead of stress as the dependent variable. Since q is a function of σ and τ^2 and since σ and τ are constants on the stress path, c_2 is constant along each stress path. The simple wave associated with c_2 therefore is in fact a circularly polarized shock wave.

(ii) $\eta = \eta_1$, $(c = c_1)$ or $\eta = \eta_3$, $(c = c_3)$

The right eigenvector $\mathbf{r}$ of eqn (20) for $\eta = \eta_1$ or η_3 is

$$\mathbf{r} = \left\{\frac{1}{2}(-h_{,\sigma} + q + 2\tau^2 q_{,\tau^2}) \pm Y,\ -s_2 q_{,\sigma}, -s_3 q_{,\sigma}\right\} \tag{36}$$

where the upper sign is for $\eta = \eta_1$ and the lower sign is for $\eta = \eta_3$. Equation (21a) becomes, making use of eqn (28),

$$\theta = \text{constant} \tag{37}$$

$$\frac{d\tau}{d\sigma} = \frac{-\tau q_{,\sigma}}{\frac{1}{2}(-h_{,\sigma} + q + 2\tau^2 q_{,\tau^2}) \pm Y} = \frac{\frac{1}{2}(-h_{,\sigma} + q + 2\tau^2 q_{,\tau^2}) \mp Y}{2\tau h_{,\tau^2}} \tag{38}$$

The second equality of eqn (38) follows from the definition of Y given in eqn (32). If we interpret eqn (28) as the transformation from the rectangular coordinate system (s_1, s_2, s_3) to the cylindrical coordinate system (σ, τ, θ), eqn (37) implies that the stress paths for $c = c_1$ or c_3

simple waves are confined to a radial plane of the cylindrical coordinates. They are called "plane polarized" simple waves. Integration of eqn (38) provides one-parameter family of curves for each of $c = c_1$ and $c = c_3$ in the (σ, τ) plane which is a radial plane.

Noticing that the upper sign is for $c = c_1$ and the lower sign is for $c = c_3$, one obtains from eqn (38) that

$$\left(\frac{d\tau}{d\sigma}\right)_{c=c_1}\left(\frac{d\tau}{d\sigma}\right)_{c=c_3} = -q_{,\sigma}/(2h_{,\tau^2}). \tag{39}$$

For hyperelastic materials the right-hand side reduces to -1 in view of eqn (29). Thus the stress paths for $c = c_1$ and $c = c_3$ are orthogonal to each other for hyperelastic materials. Taking into account of the stress paths for $c = c_2$, we see that for hyperelastic materials the stress paths for all three wave speeds are orthogonal to each other in the (s_1, s_2, s_3) space.

For some initial and boundary conditions which are constants, one may be able to obtain the solution by suitably combining the simple wave solutions associated with all three wave speeds. The circularly polarized simple wave c_2 is a circularly polarized shock wave across which the stress state (σ, τ) jumps from one radial plane to another without altering the values of (σ, τ). It follows that one would have two simple waves $c = c_1$ and c_3 and one shock wave $(c = c_2)$. However, as we will see in the next section, when the stress path passes through the stress state at which $c_1 = c_3$ one would have four simple waves.

5. SECOND ORDER ISOTROPIC HYPERELASTIC MATERIALS

For illustrative purposes we will consider isotropic hyperelastic materials of second order, i.e. the r.h.s. of eqn (27) contain s_1, s_2, s_3 of order up to two. Hence

$$h = a\sigma + \frac{b}{2}\sigma^2 + \frac{e}{2}\tau^2 \tag{40a}$$

$$q = d + e\sigma \tag{40b}$$

where a, b, d and e are material constants. The appearance of constant e in both equations ensures that the material is hyperelastic. Equation (27) now has the expression, using eqns (24) and (28),

$$\left.\begin{aligned} \epsilon &= a\sigma + \frac{b}{2}\sigma^2 + \frac{e}{2}\tau^2. \\ \gamma &= \tau(d + e\sigma). \end{aligned}\right\} \tag{41}$$

For linear materials $b = e = 0$ and hence a and d can be identified with the Lamé constants λ and μ by

$$\left.\begin{aligned} d &= 1/\mu \\ a &= 1/(\lambda + 2\mu)\ . \end{aligned}\right\} \tag{42}$$

Since Lamé constants are positive, we have

$$d > 2a > 0. \tag{43}$$

Unless otherwise stated, we exclude the linear case $b = e = 0$.

5.1 *Wave speeds*

With eqns (40), eqn (31) becomes

$$\left.\begin{aligned} \eta_1 &= \frac{1}{2}(a + b\sigma + d + e\sigma) - Y \\ \eta_2 &= d + e\sigma \\ \eta_3 &= \frac{1}{2}(a + b\sigma + d + e\sigma) + Y \\ Y &= \frac{1}{2}\{(-a - b\sigma + d + e\sigma)^2 + 4e^2\tau^2\}^{1/2}\ . \end{aligned}\right\} \tag{44}$$

It can be shown that

$$(\eta_2 - \eta_1)(\eta_3 - \eta_2) = e^2\tau^2 \geq 0. \tag{45}$$

This and the fact that $\eta_3 - \eta_1 = 2Y \geq 0$ lead to the following result

$$\eta_1 \leq \eta_2 \leq \eta_3 \tag{46}$$

or, in view of eqn (19),

$$c_1 \geq c_2 \geq c_3. \tag{47}$$

Simple waves associated with c_1 (or η_1) and c_3 (or η_3) will be called, respectively, the "fast" and "slow" simple waves. They were called "quasi-longitudinal" and "quasi-transverse" waves in [4, 5, 7]. The latter names are appropriate when σ and τ are small. As we will see shortly the slow simple waves can be longitudinal.

It is instructive to see how the wave speeds c_1 and c_3 depend on (σ, τ) by considering in the (σ, τ) plane the contour lines for constant η_1 or η_3. To this end, we substitute eqns (40) into the quadratic part of eqn (30) and rearrange the result to obtain

$$(\eta - d - e\sigma)(\eta - a - b\sigma) - e^2\tau^2 = 0. \tag{48}$$

Thus the contour lines are ellipses if $eb < 0$ and hyperbolas if $eb > 0$. In particular, the contour line for $\eta = 0$ is

$$(\sigma - m_1)(\sigma - m_2) - \frac{e}{b}\tau^2 = 0 \tag{49}$$

where

$$m_1 = -a/b, \quad m_2 = -d/e. \tag{50}$$

If $be < 0$, it is an ellipse which intersects the σ-axis at $\sigma = m_1$ and m_2 where m_1 and m_2 are on the opposite sides of $\sigma = 0$. All three wave speeds are real inside this ellipse. If $be > 0$, it is a hyperbola which also intersects the σ-axis at $\sigma = m_1$ and m_2 where m_1 and m_2 are on the same side of $\sigma = 0$. All three wave speeds are real in the region which contains the origin $\sigma = \tau = 0$ and bounded by one branch of the hyperbola which is nearer to the origin. Notice that on the σ-axis $\eta_2 = 0$ at $\sigma = m_2$ and either $\eta_1 = 0$ or $\eta_3 = 0$ at $\sigma = m_1$.

From eqns (44) and (46) we see that if $Y = 0$, $\eta_1 = \eta_2 = \eta_3$. We therefore have

$$\eta_1 = \eta_2 = \eta_3 = d + e\sigma^* = a + b\sigma^* \tag{51}$$

at

$$\tau = 0, \quad \sigma = \sigma^* = \frac{d - a}{b - e}. \tag{52}$$

The following identities between σ^*, m_1 and m_2 can be derived from eqns (50) and (52)

$$m_2 = k\sigma^* + (1 - k)m_1 \tag{53}$$

$$(\sigma^* - m_1)(\sigma^* - m_2) = (1 - k)\left(\frac{m_1 - m_2}{k}\right)^2 \tag{54}$$

where

$$k = 1 - \frac{b}{e}. \tag{55}$$

We see from eqn (54) that σ^* is located between $\sigma = m_1$ and $\sigma = m_2$ if $k > 1$ and outside of the (m_1, m_2) interval if $k < 1$.

5.2 Stress paths for simple waves

When we change the sign of e, b, σ and ϵ, eqns (41) and (55) remain the same. We will therefore consider

$$e > 0 \tag{56}$$

only, because the solution for $e < 0$ for the same k value can be obtained from the solution for $e > 0$ by changing the sign of σ and ϵ. The degenerated case $e = 0$ will be discussed separately.

The stress paths for simple waves associated with $c = c_1$ and c_3 are given by eqn (38) which reduces to, using eqns (40),

$$\left.\begin{aligned} \frac{\mathrm{d}\bar{\sigma}}{\mathrm{d}\tau} &= -\frac{k\bar{\sigma}}{2\tau} \mp \left\{\left(\frac{k\bar{\sigma}}{2\tau}\right)^2 + 1\right\}^{1/2} \\ &= \left\{\frac{k\bar{\sigma}}{2\tau} \mp \left(\left(\frac{k\bar{\sigma}}{2\tau}\right)^2 + 1\right)^{1/2}\right\}^{-1} \end{aligned}\right\} \tag{57}$$

where the $\mp$ signs are for $c = c_1$ and c_3, respectively, k is defined in eqn (55) and

$$\bar{\sigma} = \sigma - \sigma^*. \tag{58}$$

Equation (57) can be integrated in closed form if we replace $(\bar{\sigma}/\tau)$ by a new variable. In [13], eqn (57) was solved explicitly and its characteristics were discussed in detail for $k \le 0$. In the present problem k is arbitrary. Depending on the value of k, the stress paths obtained from eqn (57) can be divided into four cases and are shown in Figs. 1–4. The solid lines are the stress paths for fast simple waves while the dashed lines are for slow simple waves.

$$\left.\begin{array}{lll} \textit{Case } 1: & & e > 0, -\infty < k \le -1. \\ \textit{Case } 2: & & e > 0, -1 < k \le 0. \\ \textit{Case } 3: & \text{(a)} & e > 0, \quad 0 < k \le 1 - a/d. \\ & \text{(b)} & e > 0, \quad 1 - a/d < k \le 1. \\ \textit{Case } 4: & \text{(a)} & e > 0, \quad 1 < k \le 2. \\ & \text{(b)} & e > 0, \quad 2 < k \le 4. \\ & \text{(c)} & e > 0, \quad 4 < k < \infty. \end{array}\right\} \tag{59}$$

Notice that these cases are for $e > 0$. For $e < 0$ one simply takes the negative σ-axis as the positive σ-axis in Figs. 1–4.

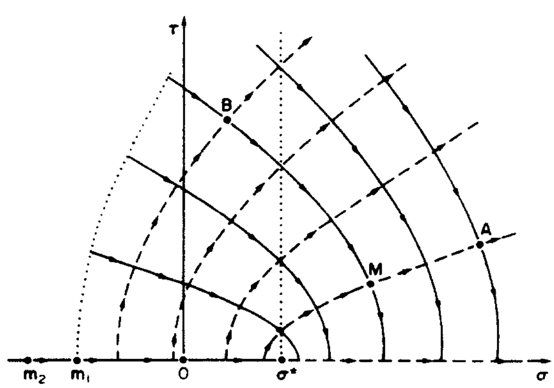

Fig. 1. Case 1: $e > 0$, $-\infty < k \le -1$. ($m_2 \to -\infty$ as $k \to -\infty$).

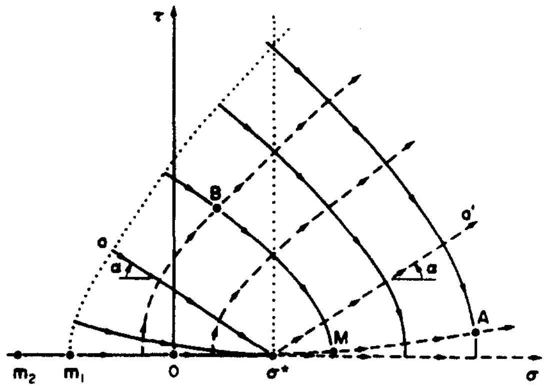

Fig. 2. Case 2: $e>0$, $-1<k\le 0$. ($\tan\alpha=(1+k)^{1/2}<1$. $\sigma^*\to\infty$ as $k\to 0$).

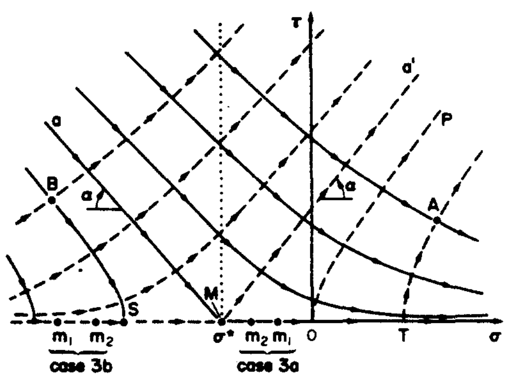

Fig. 3. Case 3a: $e>0$, $0<k\le 1-a/d$. Case 3b: $e>0$, $1-a/d<k\le 1$. ($m_1=m_2=\sigma^*$ when $k=1-a/d$, $m_1\to-\infty$ as $k\to 1$, $1<\tan\alpha=(1+k)^{1/2}\le\sqrt{2}$).

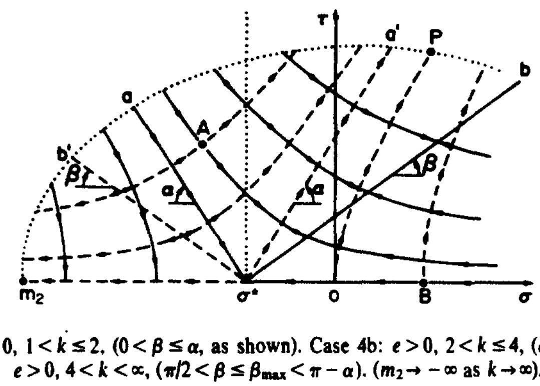

Fig. 4. Case 4a: $e>0$, $1<k\le 2$, $(0<\beta\le\alpha$, as shown). Case 4b: $e>0$, $2<k\le 4$, $(\alpha<\beta\le\pi/2)$. Case 4c: $e>0$, $4<k<\infty$, $(\pi/2<\beta\le\beta_{max}<\pi-\alpha)$. ($m_2\to-\infty$ as $k\to\infty$).

The difference between Case 3a and 3b is the location of m_1 and m_2, Fig. 3. The difference between Cases 4a, 4b and 4c will be explained later. As stated earlier, all three wave speeds are real in the region to the right of the hyperbola which passes through $\sigma=m_1$ in Figs. 1 and 2. In Fig. 3 all three wave speeds are real in the region to the right of the branch of hyperbola (not shown) which passes through $\sigma=m_1$ for Case 3a and $\sigma=m_2$ for Case 3b. For Case 4 the wave speeds are real inside the ellipse which passes through $\sigma=m_1$ and m_2. The following observations should be noted:

(i) The stress paths for the fast and slow simple waves are orthogonal to each other except at the singular point $(\sigma^*,0)$ where $c_1=c_2=c_3$, [13]. For Case 2 there are infinitely many stress paths passing through the singular point, Fig. 2.

(ii) If we replace $\bar{\sigma}$ by $-\bar{\sigma}$ in the r.h.s. of eqn (57), $(d\bar{\sigma}/d\tau)_{c=c_1}$ is identical to $-(d\bar{\sigma}/d\tau)_{c=c_3}$.

Therefore, the stress paths for the fast simple waves are the mirror image with respect to the line $\sigma = \sigma^*$ of the stress paths for the slow simple waves and vice versa. If follows from (i) that all stress paths intersect the line $\sigma = \sigma^*$ at 45°.

(iii) From eqn (57) it is seen that $d\bar{\sigma}/d\tau$ is constant along the lines $\bar{\sigma}/\tau$ = constant. This implies that all stress paths intersect any straight line draw from $(\sigma^*, 0)$ at the same angle. Hence all stress paths are similar. It is possible that one of the straight lines from $(\sigma^*, 0)$ is a stress path. To find this line, we set $d\bar{\sigma}/d\tau = \bar{\sigma}/\tau$ in eqns (57) and obtain

$$\mp \tau/\bar{\sigma} = (1 + k)^{1/2} = \tan\alpha \tag{60}$$

where the upper (or lower) sign is for c_1 (or c_3) and α is the angle this line makes with the σ-axis. We see that α exists only for $k > -1$, (see lines σ^*a and σ^*a' in Figs. 2–4).

(iv) The σ-axis is itself a stress path for simple waves and corresponds to longitudinal waves because $\tau = 0$. However, one side of $\sigma = \sigma^*$ is the stress path for the fast simple waves ($c = c_1$) while the other side is for the slow simple waves ($c = c_3$). Therefore, both the fast and slow simple waves can have longitudinal waves.

(v) Along a stress path σ is a function of τ. To determine if the wave speed increases or decreases along the stress path, we find the sign of

$$\frac{d\eta}{d\tau} = \frac{\partial\eta}{\partial\tau} + \frac{\partial\eta}{\partial\sigma}\frac{d\sigma}{d\tau} \tag{61}$$

where $\eta = \eta_1$ or η_3 is given by eqn (44) and $d\sigma/d\tau$ is given in eqn (57). In Figs. 1–4 the arrows along the stress paths indicate the direction along which c is decreasing. To find the point on a stress path at which c is a maximum or minimum, we set the r.h.s. of eqn (61) to zero. We obtain

$$\pm\frac{\tau}{\bar{\sigma}} = \frac{k(3(k-1))^{1/2}}{4-k} = \tan\beta. \tag{62}$$

These are the lines σ^*b and σ^*b' in Fig. 4 which make an angle β to the σ-axis. Equation (62) requires that $k > 1$ and hence only Case 4 can have a maximum or minimum wave speed along the stress paths. As shown in Fig. 4, when the stress path for the fast (or slow) simple waves intersects the line σ^*b (or σ^*b'), the fast (or slow) wave speed is a minimum (or maximum).

The angle β increases as k increases for Cases 4a and 4b. For Case 4a where $1 < k \le 2$, $0 < \beta \le \alpha$. For Case 4b where $2 < k \le 4$, $\alpha < \beta \le \pi/2$. For Case 4c where $k > 4$, β increases beyond $\pi/2$ as k increases, reaches a maximum angle β_{max}, and decreases to $\pi/2$ as $k \to \infty$. At the k value at which $\beta = \beta_{max}$ it can be shown that

$$\beta_{max} + \alpha < \pi. \tag{63}$$

Therefore the line σ^*b (or σ^*b') always lies to the right (or left) of the line σ^*a (or σ^*a').

One special case which is not easily reducible from the four cases mentioned so far is the case $e = 0$. We call this

$$\text{Case } 1'\!: \; e = 0, \; b > 0, \; k = -\infty. \tag{64}$$

The stress paths for this case are shown in Fig. 5. The point m_2 is at infinity. The wave speed is constant along the vertical lines regardless of whether the line is the stress path for fast or slow simple wave. If $b < 0$, Fig. 5 still applies if we take the negative σ-axis as the positive σ-axis.

5.3 *Centered simple wave solutions*

Suppose that the half space $X > 0$ is initially at rest and the stress at $t = 0$ is given by

$$(\sigma, \tau, \theta) = (\sigma^b, \tau^b, \theta^b) \tag{65}$$

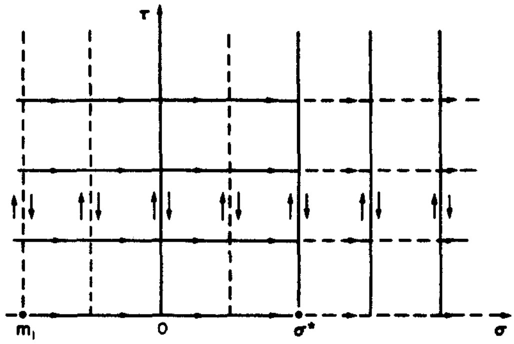

Fig. 5. Case 1′: $e = 0$, $b > 0$, $k = -\infty$. (The wave speed is constant along the vertical stress paths.)

while the stress applied at $X = 0$ is

$$(\sigma, \tau, \theta) = (\sigma^a, \tau^a, \theta^a) \tag{66}$$

where the superscript b and a stands for "before" and "after", respectively. As discussed earlier, if $\theta^a \neq \theta^b$ there is a c_2 shock wave which brings the stress state on the θ^b-radial plane to the θ^a-radial plane. In Figs. 1 and 2 let (σ^b, τ^b) and (σ^a, τ^a) be given by the points B and A, respectively. Then the solution consists of a fast simple wave in which the stress changes from point B to point M and a slow simple wave in which the stress changes from point M to point A. This is depicted in Fig. 6 in which (σ^m, τ^m) is the stress state at point M in Figs. 1 and 2. The c_2 shock wave changes the stres state (σ^m, τ^m) from the θ^b-radial plane to the θ^a-radial plane. In both Figs. 1 and 2 the stress path from B to M and then to A is in the direction of decreasing wave speed and is consistent with the description of the solution shown in Fig. 6. An admissible stress path for the simple waves is the one which follows the arrows shown in Figs. 1–5. One can shift from a fast simple wave path (solid line) to a slow simple wave path (dashed line) but not vice versa except at the singular point $(\sigma^*, 0)$ in Fig. 3. If the points B and A are given as shown in Fig. 4, there is no admissible stress path for a simple wave solution. In this case shock waves are generated. We will discuss shock waves in the following sections.

Before we close this section consider the example in which points B and A are given in Fig. 3. The admissible stress path is BSMTA and the solution is shown in Fig. 7. Notice that the stress at point M is $(\sigma^*, 0)$ and because $\tau^m = 0$ there is no c_2 shock wave. In fact, the stress

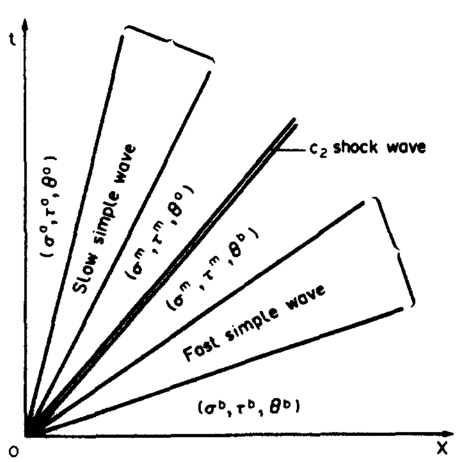

Fig. 6. Centered simple wave solution.

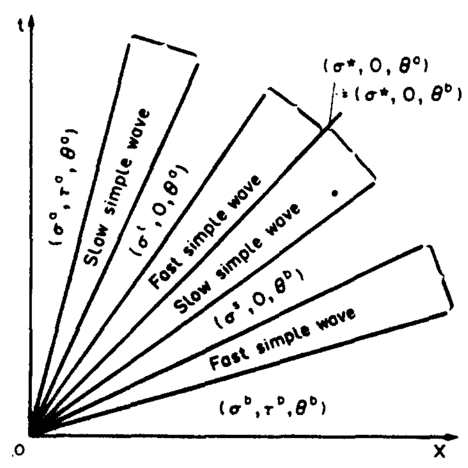

Fig. 7. The solution for $(\sigma^b, \tau^b) = B$ and $(\sigma^a, \tau^a) = A$ in Fig. 3.

state can be moved from the θ^b-plane to the θ^a-plane at any time when (σ, τ) is between the points S and T without the presence of a c_2 shock wave because $\tau = 0$. We see that we have four simple waves for this example. At point $(\sigma^*, 0)$ we have $c_1 = c_2 = c_3$. This is a singular point which always provides some unexpected phenomena [13].

6. PLANE SHOCK WAVES IN GENERAL ISOTROPIC MATERIALS

If we denote the jump of a quantity f across a shock wave by

$$[f] = f^- - f^+ \tag{67}$$

where f^- and f^+ are the values of f behind and ahead of the shock wave, the jump conditions representing the conservation of momentum and the continuity of displacements can be written as [10]

$$\left.\begin{aligned} [s_i] + \rho_0 V[v_i] = 0 \\ [v_i] + V[p_i] = 0 \end{aligned}\right\} \tag{68}$$

where V is the shock wave speed. Elimination of $[v_i]$ and making use of eqn (27) we have

$$[s_1] = \rho_0 V^2[h] \tag{69a}$$

$$[s_2] = \rho_0 V^2[s_2 q] \tag{69b}$$

$$[s_3] = \rho_0 V^2[s_3 q] \tag{69c}$$

where h and q are given functions of s_1 and $(s_2^2 + s_3^2)$. For a fixed (s_1^+, s_2^+, s_3^+), eqns (69) provide three equations for (s_1^-, s_2^-, s_3^-) with the shock wave speed V as the parameter. When V varies, (s_1^-, s_2^-, s_3^-) traces a "stress path for shock wave" in the stress space. Each point on the stress path corresponds to a shock speed V. We will see that there are three stress paths for shock waves emanating from the point (s_1^+, s_2^+, s_3^+) and that they are tangential to the stress paths for simple waves at (s_1^+, s_2^+, s_3^+).

Elimination of V between eqns (69b) and (69c) and simplifying the result, one obtains

$$[q][s_2/s_3] = 0. \tag{70}$$

Therefore either $[q] = 0$ or $[s_2/s_3] = 0$. We will discuss these two cases separately.

(1) *Plane polarized shocks*

When $[s_2/s_3] = 0$, it follows from eqn (28) that

$$\theta^- = \theta^+ \tag{71}$$

and eqns (69b) and (69c) are reduced to the same equation

$$[\tau] = \rho_0 V^2[\tau q]. \tag{72}$$

This and eqn (69a) can be combined and written as

$$\xi = \frac{[\tau q]}{[\tau]} = \frac{[h]}{[\sigma]} \tag{73}$$

where

$$\xi = (\rho_0 V^2)^{-1}. \tag{74}$$

Equation (71) indicates that the stress in front of and behind the shock wave are on the same radial plane in the (s_1, s_2, s_3) space and hence this is a plane polarized shock wave. The second equality of eqn (73) provides the stress path for shock wave in the (σ, τ) radial plane. We will prove that there are two stress paths for shock wave which are tangential to the two stress paths for simple waves at (σ^+, τ^+) and that the shock wave speed V on each stress path for shock wave approaches c_1^+ and c_3^+, respectively, as (σ^-, τ^-) approaches (σ^+, τ^+).

When (σ^-, τ^-) is very near (σ^+, τ^+), eqn (73) in the limit becomes

$$\xi = \frac{\mathrm{d}(\tau q)}{\mathrm{d}\tau} = \frac{\mathrm{d}h}{\mathrm{d}\sigma}. \tag{75}$$

Noticing that q and h are functions of σ and τ^2, we have

$$\xi = q + \tau\left(q_{,\sigma}\frac{\mathrm{d}\sigma}{\mathrm{d}\tau} + 2\tau q_{,\tau^2}\right) = h_{,\sigma} + 2\tau\frac{\mathrm{d}\tau}{\mathrm{d}\sigma}h_{,\tau^2}. \tag{76}$$

If we solve for $\mathrm{d}\tau/\mathrm{d}\sigma$ from the second equality, we find that there are two solutions and they are identical to $\mathrm{d}\tau/\mathrm{d}\sigma$ for stress paths for simple waves given by eqn (38). Substitution of $\mathrm{d}\tau/\mathrm{d}\sigma$ back to the first equality of eqn (76) and comparison with eqn (31) shows that $\xi = \eta_1$ or η_3. Therefore $V = c_1^+$ or c_3^+ at (σ^+, τ^+). This completes our proof.

We will denote by V_1 (or V_3) the shock wave speed associated with points on the stress path for shock wave which reduces to c_1^+ (or c_3^+) when (σ^-, τ^-) approaches (σ^+, τ^+).

(2) *Circularly polarized shock*

When $[q] = 0$, eqns (69b) and (69c) yield

$$\xi = (\rho_0 V^2)^{-1} = q = \text{constant} \tag{77}$$

and eqn (69a) becomes

$$h - q\sigma = \text{constant}. \tag{78}$$

Equations (77) and (78) provide two equations for (σ^-, τ^-) when (σ^+, τ^+) is known. A trivial solution is

$$[\sigma] = [\tau] = 0 \tag{79}$$

which is identical to eqn (35) and the stress path for shock wave is a circle in the (s_1, s_2, s_3) space. Denoting the shock wave speed given in eqn (77) by V_2 and using eqn (31), we have

$$V_2 = c_2^+ = c_2^-. \tag{80}$$

If eqns (77) and (78) yield a solution other than eqn (79), we would have a shock wave with $V = c_2$ which is not circularly polarized. However, a shock wave of this kind can be regarded as two shock waves merged into one; one circularly polarized shock wave in which the stress jumps from one radial plane to another and one plane polarized shock wave in which the stress jumps from (σ^+, τ^+) to (σ^-, τ^-).

It should be noted that not every point on the stress path for shock wave is an admissible shock. According to Lax[14], a shock wave is stable if

$$c_i^- \geq V_i \geq c_i^+, \quad (i = 1, 2, 3). \tag{81}$$

Otherwise, the shock wave may develop into a simple wave. Equation (80) implies that eqn (81) is satisfied for $i = 2$ and hence the shock wave associated with V_2 is always stable and admissible. However, shock waves associated with V_1 and V_3 may be unstable unless eqn (81) for $i = 1, 3$ are satisfied. To indicate the dependence of V_i and c_i on (σ^+, τ^+) and (σ^-, τ^-), we may also write eqn (81) as

$$c_i(\sigma^-,\tau^-) \geq V_i(\sigma^-,\tau^-;\sigma^+,\tau^+) \geq c_i(\sigma^+,\tau^+) \tag{82}$$

where it is clear that

$$V_i(\sigma^+,\tau^+;\sigma^-,\tau^-) = V_i(\sigma^-,\tau^-;\sigma^+,\tau^+). \tag{83}$$

7. STRESS PATHS FOR SHOCK WAVES IN SECOND ORDER ISOTROPIC HYPERELASTIC SOLIDS

The stress paths for shock waves V_1, V_3 for second order isotropic hyperelastic materials can be written as, using eqns (73) and (40)

$$\xi = \frac{d[\tau] + e[\sigma\tau]}{[\tau]} = \frac{a[\sigma] + \frac{b}{2}[\sigma^2] + \frac{e}{2}[\tau^2]}{[\sigma]}. \tag{84}$$

If $\tau^+ \neq 0$, the second equality can be rewritten as

$$\left\{1 + \frac{k+1}{2\tau^+}[\tau]\right\}\frac{[\sigma]^2}{[\tau]^2} + k\frac{\bar{\sigma}^+}{\tau^+}\frac{[\sigma]}{[\tau]} - \left(1 + \frac{[\tau]}{2\tau^+}\right) = 0 \tag{85}$$

or

$$\frac{[\sigma]}{[\tau]} = \frac{-\frac{k\bar{\sigma}^+}{2\tau^+} \mp \left\{\left(\frac{k\bar{\sigma}^+}{2\tau^+}\right)^2 + \left(1 + \frac{[\tau]}{2\tau^+}\right)\left(1 + \frac{k+1}{2\tau^+}[\tau]\right)\right\}^{1/2}}{1 + \frac{K+1}{2\tau^+}[\tau]} \tag{86}$$

where $\bar{\sigma}^+ = \sigma^+ - \sigma^*$ and the $\mp$ signs are for V_1 and V_3, respectively. This provides the stress path for admissible (σ^-,τ^-) when (σ^+,τ^+) is given provided that the stability condition, eqn (82), is satisfied. Notice that as $[\sigma] \to 0$ and $[\tau] \to 0$, eqn (86) reduces to eqn (57). Notice also that, unlike the stress path for simple waves, one cannot choose any point on the stress path for shock wave as a new (σ^+,τ^+). If one chooses any point on a stress path for shock wave as a new (σ^+,τ^+), eqn (86) in general yields a new stress path for this new choice of (σ^+,τ^+).

If $\tau^+ = 0$, the second equality of eqn (84) yields

$$\tau^- = 0 \tag{87a}$$

or

$$(\tau^-)^2 = (k+1)[\sigma]\left([\sigma] + \frac{2k}{1+k}\bar{\sigma}^+\right). \tag{87b}$$

Equation (87a) is the equation for the σ-axis while eqn (87b) is a hyperbola. Moreover, eqn (84) yields

$$\xi = d + e\sigma^- = \eta_2^-, \quad \text{when } \tau^- \neq \tau^+ = 0 \tag{88a}$$

$$\xi = a + b(\sigma^+ + \sigma^-)/2, \quad \text{when } \tau^- = \tau^+ = 0 \tag{88b}$$

where use has been made of eqn (44).

A particular case of eqn (87b) which will be useful later on is when $\sigma^+ = 0$. We have

$$(\tau^-)^2 = (k+1)\sigma^-\left\{\sigma^- - \frac{2k}{1+k}\sigma^*\right\}. \tag{89}$$

One branch of this hyperbola intersects the σ-axis at $\sigma = \hat{\sigma}$ where

$$\hat{\sigma} = \frac{2k}{1+k}\sigma^*. \tag{90}$$

For q and h given by eqns (40), eqns (77) and (78) have no solutions other than eqn (79). Hence for second order isotropic hyperelastic materials the V_2 shock wave is always a circularly polarized shock wave.

8. SOLUTION OF INITIAL AND BOUNDARY VALUE PROBLEMS

In Section 5.3 we considered certain initial and boundary value problems for which the solutions can be obtained by suitably combining the fast (c_1) and slow (c_3) simple waves as well as the circularly polarized simple wave c_2 which can also be regarded as the circularly polarized shock wave V_2. For other initial and boundary value problems one has to introduce the shock waves V_1 and/or V_3. For illustrative purposes we will consider second order isotropic hyperelastic materials and the following initial and boundary conditions

$$\left.\begin{aligned}(\sigma, \tau) &= (0, 0), \quad &\text{at} \quad t = 0\\ &= (\sigma^a, \tau^a), \quad &\text{at} \quad X = 0\end{aligned}\right\}. \tag{91}$$

Depending on the location of (σ^a, τ^a) in the (σ, τ) plane, the solution may consist of a different combination of simple waves and/or shock waves. However, with the stress paths for simple waves and shock waves introduced in this paper, one can determine the right combination for the solution. We will use Case 3b and Case 4a to illustrate our point.

8.1 *Case* 3*b*. $e > 0$, $1 - a/d < k \le 1$

With the initial value being zero and the boundary value prescribed as (σ^a, τ^a), the solution depends on the position of (σ^a, τ^a) in the (σ, τ) plane, Fig. 8 (see also Fig. 3). If (σ^a, τ^a) is the point A_1 in region I bounded by the positive σ-axis and OP which is the stress path for slow simple wave from the origin, we have the solution which consists of two simple waves as shown in Fig. 9. If (σ^a, τ^a) is outside of region I, there is no solution which consists of simple waves only. In this case we consider a shock wave V_1 with $(\sigma^+, \tau^+) = (0, 0)$. The stress path for shock wave V_1 is the negative σ-axis $O\hat{\sigma}m_2$ as given by eqn (87a) and the curve $\hat{\sigma}Q$ which is one branch of the hyperbola given by eqn (89). The other branch of the hyperbola violates the stability condition, eqn (82), and hence is not admissible. The point $\hat{\sigma}$ is given by eqn (90). It can be shown that when (σ^-, τ^-) is on $O\hat{\sigma}m_2$ or $\hat{\sigma}Q$ the shock wave is stable, i.e.

$$c_1(\sigma^-, \tau^-) \ge V_1(\sigma^-, \tau^-; 0, 0) \ge c_1(0, 0). \tag{92}$$

We use double solid lines (or double dashed lines) to denote the stress path for admissible shock wave V_1 (or V_3). The arrows on the path indicate the direction from $(0, 0)$ to (σ^-, τ^-).

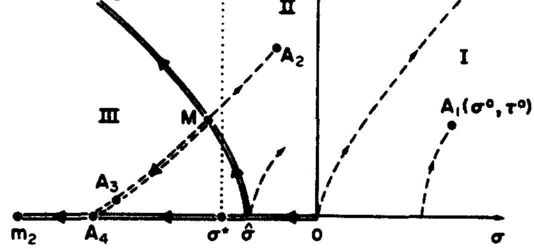

Fig. 8. Case 3b: Initial condition $(\sigma, \tau) = (0, 0)$, boundary condition (σ^a, τ^a) arbitrary.

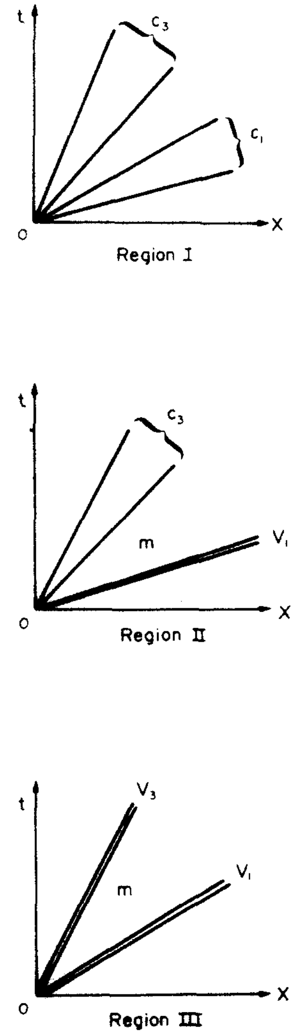

Fig. 9. Case 3b: Solution in the (X, t) plane for (σ^a, τ^a) in the region shown in Fig. 8.

Thus if (σ^a, τ^a) is the point A_2 in region *II* bounded by $Q\hat{\sigma}OP$, the solution consists of a shock wave V_1 which carries the stress from point O to M followed by a slow simple wave c_3 which carries the stress to A_2. Since the slow simple wave follows the shock wave V_1, Fig. 9, we must have

$$c_3(\sigma^m, \tau^m) \leq V_1(\sigma^m, \tau^m; 0, 0) \tag{93}$$

where (σ^m, τ^m) is the stress in the constant state region m. It can be shown that (93) is satisfied for (σ^m, τ^m) on $O\hat{\sigma}Q$ in Fig. 8. It can also be shown that the equality in (93) holds at $(\sigma^m, \tau^m) = (\hat{\sigma}, 0)$. Thus if point A_2 were on the stress path for slow simple wave from point $\hat{\sigma}$ in Fig. 8, the constant state region m in Fig. 9 vanishes. The simple wave c_3 follows the shock wave V_1 without a constant region. Equation (93) is violated for (σ^m, τ^m) which lies between the points $\hat{\sigma}$ and σ^*. Hence a slow simple wave c_3 cannot follow a shock wave which carries the stress from $(0, 0)$ to a point between $\hat{\sigma}$ and σ^*.

If (σ^a, τ^a) is the point A_3 in region *III*, Fig. 8, which is bounded by $\hat{\sigma}m_2$ and $\hat{\sigma}Q$ the solution consists of shock waves V_1 and V_3, Fig. 9. In Fig. 8 the double dashed line MA_4 is the stress path for shock wave V_3 which is obtained from eqn (86) by setting $(\sigma^+, \tau^+) = (\sigma^m, \tau^m)$. It can be shown that when (σ^a, τ^a) is on MA_4, the shock wave is stable

$$c_3(\sigma^a, \tau^a) \geq V_3(\sigma^a, \tau^a; \sigma^m, \tau^m) \geq c_3(\sigma^m, \tau^m). \tag{94}$$

For both shock waves to exist as shown in Fig. 9, we must have

$$V_3(\sigma^a, \tau^a; \sigma^m, \tau^m) \le V_1(\sigma^m, \tau^m; 0, 0). \tag{95}$$

This is satisfied for (σ^a, τ^a) in region *III*.

Notice that there are actually more than three types of solutions as shown in Fig. 9. For instance, if (σ^a, τ^a) is on the positive σ-axis (or on OP), the solution consists of one simple wave c_1 (or c_3) only. Similarly, if (σ^a, τ^a) is on the negative σ-axis or on $\hat{\sigma}Q$, the solution consists of one shock wave V_1 only.

8.2 *Discontinuous dependence of solution on boundary conditions*

The stress path MA_4 for shock wave V_3 in Fig. 8 is obtained from eqn (86) by letting $(\sigma^+, \tau^+) = (\sigma^m, \tau^m)$. The path intersects the σ-axis at point A_4; the location of A_4 can be determined by letting $\tau^- = 0$ to find σ^- in eqn (86). If (σ^a, τ^a) is at point A_3, we have the solution in which a V_1 shock wave carries the stress from point O to M followed by a V_3 shock wave which carries the stress from M to A_3. On the other hand, if (σ^a, τ^a) is at point A_4, we have the solution in which only one shock wave V_1 carries the stress from point O to A_4. As point A_3 approaches A_4, it can be shown that

$$V_1(M; O) = V_3(A_4; M) = V_1(A_4; O). \tag{96}$$

The constant region m, Fig. 9, between the two shocks diminishes to zero and the two shocks coalesce into one. Therefore, in the limit as A_3 approaches A_4 the two solutions are identical. However, as long as A_3 is not on the σ-axis, we have a constant stress region m in which the shear stress τ^m is *finite*; although the constant region may be small for small time t.

Thus for a half-space which is initially stress free, an impact at the boundary $X = 0$ with $\sigma^a < \hat{\sigma}$ and $\tau^a = 0$ will produce one shock wave which is a longitudinal shock because no shear stress is generated. If τ^a is nonzero but very *small*, one has two shock waves (neither of them is longitudinal) with a *finite* shear stress between the two shocks although the region between the two shock waves may be small for small time. In experiments this implies that a slight misalignment of the longitudinal impact at $X = 0$ can produce quite different response at $X \neq 0$.

8.3 *Case 4a.* $e > 0$, $1 < k \le 2$

We choose Case 4a because it is far more complicated than Case 3b. Referring to Figs. 4 and 10, we see that if (σ^a, τ^a) is in region *I* bounded by the positive σ-axis and OP which is the slow simple wave path from the origin O, we have the solution in which a shock wave V_1 is followed by a slow simple wave c_3, Fig. 11. If (σ^a, τ^a) is in region *II* bounded by $a'\sigma^*OP$ where σ^*a' is the slow simple wave path from point σ^*, the solution consists of one fast simple wave c_1 and one slow simple wave c_3.

We will refer to Figs. 10 and 11 for the remaining regions. For (σ^a, τ^a) in region *III* which is bounded by $RD\sigma^*a'$ where σ^*D is a straight line which we will determine later and DR is the slow simple wave path from D, the solution consists of a fast simple wave c_1 from point O to E, immediately followed by a shock wave V_1 from E to M and a slow simple wave from M to $(\sigma^a,$

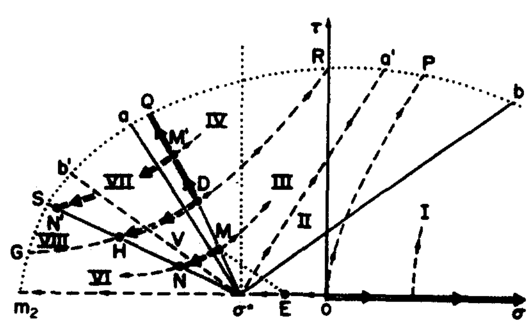

Fig. 10. Case 4a: Initial condition $(\sigma, \tau) = (0, 0)$, boundary condition (σ^a, τ^a) arbitrary.

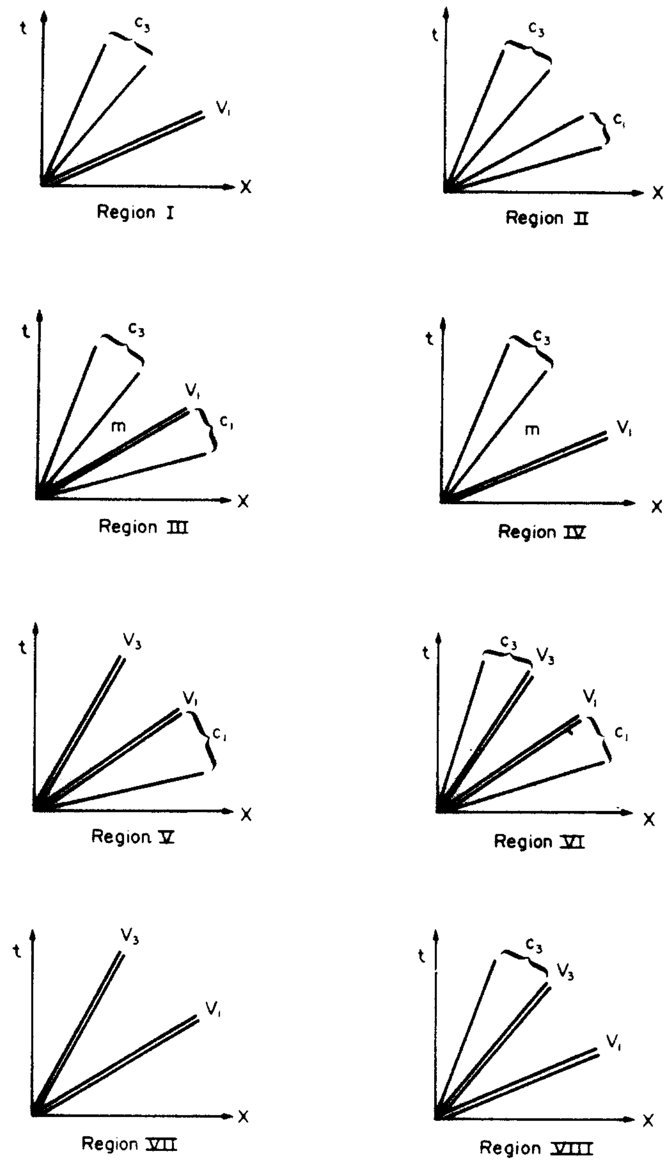

Fig. 11. Case 4a: Solution in the (X, t) plane for (σ^a, τ^a) in the region shown in Fig. 10.

τ^a). To find the relation between (σ^m, τ^m) at M and $(\sigma^e, 0)$ at E, we use eqns (88a), (44), (82) and (87b) to obtain

$$\xi_1(\sigma^m, \tau^m; \sigma^e, 0) = d + e\sigma^m \tag{97}$$

$$\eta_1(\sigma^e, 0) = a + b\sigma^e \tag{98}$$

$$c_1(\sigma^m, \tau^m) \geq V_1(\sigma^m, \tau^m; \sigma^e, 0) = c_1(\sigma^e, 0) \tag{99}$$

$$(\tau^m)^2 = (k+1)(\sigma^m - \sigma^e)\left\{(\sigma^m - \sigma^e) + \frac{2k}{1+k}(\sigma^e - \sigma^*)\right\} \tag{100}$$

where $\xi_1 = (\rho_0 V_1^2)^{-1}$ and $\eta_1 = (\rho_0 c_1^2)^{-1}$. The equality in the second of (99) is due to the fact that the shock wave V_1 follows immediately behind the fast simple wave c_1, Fig. 11. Substitution of eqns (97) and (98) into the second of eqn (99) yields

$$\sigma^m = k\sigma^* - (k-1)\sigma^e \tag{101}$$

and eqn (100) provides

$$\tau^m = k(k-1)^{1/2}(\sigma^e - \sigma^*). \tag{102}$$

As point E moves from point σ^* to O, M moves from σ^* to D. To find the line σ^*D which is the locus of M, we eliminate σ^e between eqns (101) and (102). We have

$$\frac{\tau^m}{\sigma^m - \sigma^*} = -\frac{k}{(k-1)^{1/2}} \tag{103}$$

and hence σ^*D is a straight line. The lines σ^*a and σ^*b' in Fig. 4 are reproduced in Fig. 10 to show the relative position of σ^*D. It can be shown that for (σ^a, τ^a) in region *III*, (99) is satisfied and that

$$V_1(\sigma^e, 0; \sigma^m, \tau^m) \geq c_3(\sigma^m, \tau^m). \tag{104}$$

When (σ^a, τ^a) is on the slow simple wave path *DR*, the solution consists of a shock wave V_1 from point O to D and a slow simple wave c_3 from D to (σ^a, τ^a). In Fig. 10, *DQ* is a portion of the shock wave path V_1 which is a hyperbola with $(\sigma^+, \tau^+) = (0, 0)$ given by eqn (89). It can be shown that the straight line σ^*D is tangent to the hyperbola *DQ* at point D. For (σ^a, τ^a) in region IV which is bounded by *QDR*, the solution consists of a shock wave V_1 from point O to a point on *DQ* followed by a slow simple wave c_3 to (σ^a, τ^a).

Region *V* is the triangular region σ^*DH where *DH* is the shock wave path for V_3 with stress at D as (σ^+, τ^+) and is given by eqn (86) while σ^*H is a straight line we will determine shortly. Region *VI* is bounded by $m_2\sigma^*HG$ where *HG* is the slow simple wave path from H. For (σ^a, τ^a) in region *V*, the solution consists of a fast simple wave from point O to E, a shock wave V_1 from point E to M and a shock wave V_3 from M to (σ^a, τ^a). If (σ^a, τ^a) is in region *VI*, we will have a fast simple wave from point O to E, a shock wave V_1 from point E to M, a shock wave V_3 from M to N which is on the boundary $H\sigma^*$ between regions *V* and *VI*, and a slow simple wave c_3, Fig. 11. Thus point N has the following property

$$c_3(\sigma^n, \tau^n) = V_3(\sigma^n, \tau^n; \sigma^m, \tau^m) \tag{105}$$

which means $\eta_3 = \xi_3$. This leads to, using the first equality of eqn (84) and eqn (44)

$$\left\{1 + k\frac{[\tau]}{\tau^+}\right\}\frac{[\sigma]^2}{[\tau]^2} + k\frac{\bar{\sigma}^+}{\tau^+}\frac{[\sigma]}{[\tau]} - \left(1 + \frac{[\tau]}{\tau^+}\right)^2 = 0 \tag{106}$$

where $(\sigma^+, \tau^+) = (\sigma^m, \tau^m)$, $(\sigma^-, \tau^-) = (\sigma^n, \tau^n)$. The shock path *MN* is given by eqn (85). Equations (85) and (106) provide two equations for (σ^-, τ^-) when (σ^+, τ^+) is known. If we subtract eqn (106) from (85), we have

$$\frac{[\sigma]^2}{[\tau]^2} = \frac{1}{k-1}\left(1 + \frac{2\tau}{\tau^+}\right). \tag{107}$$

Since the left-hand side is always positive, we must have $k > 1$. Therefore, eqns (85) and (106) have no solutions in other cases except Case 4. In other words, one cannot find (σ^m, τ^m) and (σ^n, τ^n) such that eqn (105) holds in Cases 1–3. The same conclusions hold if c_3 and V_3 in eqn (105) are replaced by c_1 and V_1.

As point M moves from σ^* to D, N moves from σ^* to H. Since σ^*D is a straight line, it is not difficult to show that σ^*H is also a straight line. For (σ^a, τ^a) in region *V* or *VI* it can be shown that the shock waves V_1 and V_3 are stable and that $V_3 < V_1$.

Region *VII* is bounded by *SHDQ* where *HS* is a curve which has the similar property as the line σ^*H. Thus the curve $M'N'$ in Fig. 10 is a shock wave path V_3 with the stress at M' as (σ^+, τ^+). The stress at N' satisfies eqn (105) if we replace n and m by n' and m'. Therefore, the locus of N', which is the curve *HS*, is obtained from eqns (85) and (106). Unlike σ^*H which is a straight line, HS is not a straight line because DQ is not. For (σ^a, τ^a) in region VII, the solution consists of two shock waves V_1 and V_3, Fig. 11. For (σ^a, τ^a) in region $VIII$, the solution consists of two shock waves V_1 and V_3 followed by a slow simple wave c_3. Again we can show that both shock waves are stable and that $V_3 < V_1$.

As we emphasized before, there are actually more than eight possible solutions shown in Fig. 11 because degenerated solutions are obtained when (σ^a, τ^a) is on the boundary of a region.

9. CONCLUDING REMARKS

We have demonstrated in this paper how one may use the stress paths for simple waves and shock waves to determine the right combination of simple waves and/or shock waves to satisfy given initial and boundary conditions. It should be pointed out that one could extend the idea and introduce the "velocity paths" for simple waves and shock waves in the velocity space if the initial and boundary values are prescribed for the velocity. If the deformation gradients are prescribed as the initial and boundary conditions, albeit uncommon, one could either introduce the "deformation gradient paths" or replace the prescribed deformation gradients by the stresses through the constitutive equations, eqn (23), and employ the analyses presented here.

10. ADDITIONAL REMARKS

In the analyses the region of validity in the (σ, τ) plane is based on the condition that all three wave speeds are real. For the second order hyperelastic material which we used as an illustrative example, one should also add the condition that the volume change $\|\mathbf{F}\| > 0$. From eqns (2), (24) and (41) this condition leads to

$$1 + a\sigma + \frac{b}{2}\sigma^2 + \frac{e}{2}\tau^2 > 0. \tag{108}$$

Depending on the values of the material constants a, b and e, the condition (108) may or may not reduce the region of validity based on the condition of real wave speeds. Therefore, the general procedure of obtaining a solution and the unexpected results presented here remain valid.

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